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Quotient of a Consistent Subset by an Element - Constructive Point of View

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Abstract

Consistent subset of a semigroup is important subset of semigroups in Bishop's constructive mathematics. In this article we introduce notion of quotient $[A : x]$ of consistent subset A by an element x of commutative semigroup and research its properties. Besides, we give (without the axiom of choose) a construction of the maximal strongly extensional consistent subset $C(x)$ of semigroup for any element x of semigroup such that $x \bowtie C(x)$. At the end of this investigation, we give some applications.

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1 Introduction and preliminaries

This investigation is in Semigroup Theory within Constructive Algebra, in sense of the book [1], [3] and [11] and papers [5]-[10]. Constructive Mathematics is developed on Constructive Logic (or Intuitionistic Logic) - logic without the Law of Excluded Middle: $P \vee \neg P$. We have to note that 'the crazy axiom' $\neg P \Longrightarrow (P \Longrightarrow Q)$ is included in the Constructive Logic. Precisely, in Constructive Logic the 'Double Negation Law' $P \Longleftrightarrow \neg\neg P$ does not hold, but the following implication $P \Longrightarrow \neg\neg P$ holds even in Minimal Logic. In

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Constructive Logic 'Weak Law of Excluded Middle' $\neg P \vee \neg\neg P$ does not hold. It is interesting that in Constructive Logic the following deduction principle $A \vee B, \neg A \vdash B$ holds, but this is impossible to prove without 'the crazy axiom'. An advantage of working in this manner is that proofs and results have more interpretations. On one hand, Bishop's Constructive Mathematics is consistent with the traditional mathematics. On the other hand, the results can be interpreted recursively or intuitively. If we are working constructively, the first problem is to obtain appropriate substitutes of the classical definitions.

Throughout this paper, $S = (S, =, \neq, \cdot)$ always denotes a commutative semigroup with apartness in the sense of the books [3], [4], [11] and papers [5]-[10]. The apartness "$\neq$ " on S is a binary relation with the following properties: For every elements x, y and z in S, it holds:

$$\neg(x \neq x),\ x \neq y \Longrightarrow y \neq x,\ x \neq y \wedge y = z \Longrightarrow x \neq z,$$
$$x \neq z \Longrightarrow (\forall t \in S)(x \neq t \vee t \neq z).$$

It takes that the semigroup operation is strongly extensional, in the following sense

$$(\forall a, b, x, y \in S)((ay \neq by \Longrightarrow a \neq b) \wedge (xa \neq xb \Longrightarrow a \neq b)).$$

Example 1: (1) Let $\wp(X)$ be power-set of set X. If for subsets A, B of X we define $A \neq B$ if and only if $(\exists a \in A)\neg(a \in B)$ or $(\exists b \in B)\neg(b \in A)$, then the relation "$\neq$ " is diversity relation on $\wp(X)$ but it is not an apartness.
(2) ([4]) The relation $\neq$, defined on the set $\mathbf{Q}^N$ by

$$f \neq g \Longleftrightarrow (\exists k \in N)(\exists n \in N)(m \geq n \Longrightarrow |f(m) - g(m)| > k^{-1}),$$

is an apartness on $\mathbf{Q}^N$. ♦

Let T be a subset of S. We say that it is:
(i) *Strongly extensional subset* ([1], [3]) iff $(\forall x, y \in S)(x \in T \Longrightarrow x \neq y \vee y \in T)$;
(ii) *Consistent subset* ([2]) iff $(\forall x, y \in S)(xy \in T \Longrightarrow x \in T \wedge y \in T)$;
(iii) *Prime subset* of S iff $(\forall x, y \in S)(x \in T \wedge y \in T \Longrightarrow xy \in T)$;
(iv) *Semiprime subset* of S iff $(\forall x \in S)(x \in T \Longrightarrow x^2 \in T)$;
(v) *Potent-semiprime subset* of S iff $(\forall x \in S)(\forall n \in N)(x \in T \Longrightarrow x^n \in T)$.

If T is a subset of S, we define *coradical* of T by $cr(T) = \{x \in T : (\forall n \in N)(x^n \in T)\}$. Strongly extensional consistent prime (semiprime, potent-semiprime) subset T of S is a *filter* (*semi filter*, *potent-semifilter*) of S. It is easy to show that $cr(T) \subseteq T$ and $cr(T) = T$ if and only if T is a potent semifilter of S. For a subset T we say that it is *primary* if the implication $x \in T \wedge y \in cr(T) \Longrightarrow xy \in T$ holds. Besides, if T is primary, then $cr(T)$ is a filter of S. Finally, if T is a strongly extensional consistent subset of S, then $\cup F \subseteq cr(T)$ holds, where the union is over all filters of S under T.

Example 2: Let S be the set

$$\left\{ \begin{pmatrix} 0\,0 \\ x\,x \end{pmatrix}, \begin{pmatrix} 0\,x \\ 0\,0 \end{pmatrix}, \begin{pmatrix} 0\,0 \\ 0\,x \end{pmatrix}, \begin{pmatrix} x\,x \\ 0\,0 \end{pmatrix}, \begin{pmatrix} 1\,0 \\ 0\,1 \end{pmatrix} : x \in R \wedge 0 \leq x \leq \frac{1}{2} \right\}.$$

The operation on S is the usual matrix multiplication. Then S is a semigroup with apartness. The set $Q = \{f \in S : f \neq 0\}$ is a consistent subset of S and the set

$$\left\{ \begin{pmatrix} 0\,0 \\ x\,x \end{pmatrix}, \begin{pmatrix} x\,x \\ 0\,0 \end{pmatrix}, \begin{pmatrix} 0\,0 \\ 0\,x \end{pmatrix}, \begin{pmatrix} 1\,0 \\ 0\,1 \end{pmatrix} : x \in R \wedge 0 \leq x \leq \frac{1}{2} \right\}$$

is $cr(Q)$. Let us note that $cr(Q)$ is not a consistent subset of S: if $x \neq 0$ and $y \neq 0$, then

$$cr(Q) \ni \begin{pmatrix} 0\,0 \\ 0\,xy \end{pmatrix} = \begin{pmatrix} 0\,0 \\ y\,y \end{pmatrix} \cdot \begin{pmatrix} 0\,x \\ 0\,0 \end{pmatrix}.\blacklozenge$$

Let x be an element of S and A a subset of S. We write $x \bowtie A$ iff $(\forall a \in A)(x \neq a)$, and $A' = \{x \in S : x \bowtie A\}$ (A' is the strong compliment of A). For a subset A of semigroup S we shall say that it is *detachable* in S if and only if

$$(\forall x \in S)(x \in T \vee \neg(y \in T)).$$

Remark 1.1 It is easy to show that if A is a consistent subset of semigroup S, then A' is an ideal of S. Indeed, let $x \in A' \vee y \in A'$ and let a be an arbitrary element of A. Then, we have $a \neq xy$ or $xy \in A$ by strongly extensionality of A. In the second case we have conclusion $x \in A \wedge y \in A$, which is impossible because $x \bowtie A$ or $y \bowtie A$. So, we conclude $xy \bowtie A$ because $a \neq xy$ holds for every a in A, i.e. the set A' is an ideal of S. Let us note that the opposite assertion "If J is an ideal of semigroup S then J' is a consistent subset if S" is not valid in general case.

At first, semigroup with apartness was defined and studied by Heyting. After that, several authors have worked on this important topic, as for example: Mines ([4]) Richman ([4]), Troelstra and van Dalen ([11]), and the author (artickles [5]-[10]).

For undefined notions and notations in Constructive mathematics we refer to [1], [3], [4], [5]-[10] and [11], and in the Semigroup Theory we refer to the book [2].

Let q be a relation on semigroup S. For q we say that it is a *coequality relation* if and only if it is consistent, symmetric and cotransitive relation on S

$$q \subseteq \neq,\ q^{-1} = q,\ q \subseteq q * q,$$

where "$*$ " is the *filled product* of relations (see, for example [5] or [8]) If the coequality relation q is compatible with the semigroup operation in the following sense

$$(\forall x, y, a, b \in S)((xay, ayb) \in q \Longrightarrow (x, y) \in q),$$

we say that q is *anticongruence* on S.

Consistent subsets of semigroups are very important in the semigroup theory. As a matter of fact, we have the following very interesting proposition in which we give a construction of a coequality relation on semigroup using of a consistent subset of semigroup:

Lemma 1.2 *Let A be a strongly extensional consistent subset of S. Then the relation q_A on S, defined by*

$$(x, y) \in q_A \Longleftrightarrow x \neq y \wedge (x \in A \vee y \in A),$$

is an anticongruence on S such that

$$x \in A \Longrightarrow xq = \{y \in S : y \neq x\},\ \neg(x \in A) \Longrightarrow xq = A.$$

Notes 1.3
(1) Firstly, if q is a coequality on semigroup S, then the subset $aq = \{b \in S : (a, b) \in q\}$ $(a \in S)$ is a strongly extensional subset of S. Indeed, let x be an element of S such that $x \in aq$ and let y be an arbitrary element of S. Then, from $(a, x) \in q$ we conclude $(a, y) \in q$ or $(y, x) \in q$ by cotransitivity of q. So, we have that $y \in aq$ or $x \neq y$ by consistency of relation q.

(2) If we want to have strongly extensional consistent classes of anticongruence, we need another condition: Let q be anticongruence on semigroup S with apartness. Then, classes aq $(a \in S)$ are strongly extensional consistent subsets of S if and only if $(\forall a, b \in S)((ab, a) \bowtie q)$.
Let the condition $(\forall a, b \in S)((ab, a) \bowtie q)$ holds. Let xy be an element of aq, i.e. $(xy, a) \in q$. Thus, out of $(xy, x) \in q$ or $(x, a) \in q$ and $(xy, y) \in q$ or $(y, a) \in q$, we conclude that $x \in aq$ and $y \in aq$. So, the subset aq is a consistent subset of S. Opposite to the previous, let us suppose that the class aq is a consistent subset of S for every a in S. Let (u, v) be an arbitrary element of q. Then, out of $(u, ab) \in q$ or $(ab, a) \in q$ or $(a, v) \in q$ it follows $(u, v) \neq (ab, a)$ or the implication $ab \in aq \Longrightarrow a \in aq \wedge b \in aq$ holds. Last implication is impossible because $a \bowtie aq$ holds. Hence, we have $(u, v) \neq (ab, a)$, i.e. the condition $(\forall a, b \in S)((ab, a) \bowtie q)$ holds.

2 The quotient of consistent subset by an element

Our first step in this paper is defining, in the Semigroup Theory, a new notion - notion of quotient of consistent subset by an element - and sign for this notion:

Definition 2.1 Let A be a consistent strongly extensional subset of S and $X \subseteq S$. The quotient of A by subset X is $[A : X] = \{a \in S : (\exists x \in X)(ax \in A)\}$. For $X = \{x\}$ we put $[A : x]$ instead of $[A : \{x\}]$. In this case we say that $[A : x]$ is the *quotient* of strongly extensional subset A by an element x of S.

In the following assertions 2.2-2.7, we will describe some properties of those subsets.

Theorem 2.2
(0) $[A : X] \subseteq A$
(1) $X \cap A = \emptyset \Longrightarrow [A : X] = \emptyset$;
(2) $\emptyset \neq [A : X] \iff (AX \neq \emptyset \wedge AX \cap A \neq \emptyset)$;
(3) $[A : X] \neq A \iff (\exists b \in A)(\exists x \in X)(xb \bowtie A)$;
(4) *The set* $[A : X]'$ *is an ideal of semigroup* S *and*

$$[A : X]' = \{a \in S : aX \subseteq A\}(= (A : X))$$

holds.

Proof: (4) At first, the subset $[A : X]'$ is an ideal of subset S by Remark 1.1. Secondly, let a be an element of $(A' : X)$, i.e. let $aX \subseteq A'$ and let u be an arbitrary element of $[A : X]$. Then, there exists an element x of X such that $ux \in A$ (and $ax \bowtie A$). On the other hand, as the subset A is a strongly extensional subset of S we have $ux \neq ax \vee ax \in A$. So, it has to be $ux \neq ax$ because of $ax \bowtie A$. Therefore, $u \neq a$, i.e. $a \bowtie [A : X]$. Hence, $(A' : X) \subseteq [A : X]'$. Opposite to the previous, let $a \bowtie [A : X]$ hold and let v be an arbitrary element of A. Then, we have $v \neq ax \vee ax \in A$ for each x of X. Out of $ax \in A$, we conclude $a \in A \wedge x \in A$, which is impossible. So, we have $aX \subseteq A'$. $\square$

Theorem 2.3 *Let* A *and* B *be strongly extensional consistent subsets of semigroup* S *and* $x \in S$, $X \subseteq S$. *Then:*
(5) $[A : X]$ *is a strongly extensional consistent subset of* S;
(6) $A \subseteq B \Longrightarrow [A : X] \subseteq [B : X]$;
(7) *For family* $\{B$: B *is strongly extensional consistent subsets of* $S\}$ *holds*

$$\cup[B : X] \subseteq [\cup B : X];$$

(8) $(\forall n \in N)([A : x^{n+1}] \subseteq [A : x^n] \subseteq [A : x] \subseteq A)$;
(9) *If* A *is a semifilter of* S, *then* $A = \cup\{[A : x] : x \in A\}$ *holds; and*
(10) *If* A *is a (potent) semifilter of* S *(and* n *a natural number), then*

$$[A : x^2] = [A : x] \; ([A : x^n] = [A : x])$$

holds.

Proof: (5) Let a be an element of $[A : X]$ and let b be an arbitrary element of S. Then, there exists an element x of X such that axA. As A is a strongly extensional subset of S, we have $ax \neq bx$ or $bx \in A$. Hence, $a \neq b$ or $b \in [A : X]$. Let $ab \in [A : X]$, i.e. let there exist an element x of X such that $abx \in A$. As A is a consistent subset of S, we have $ax \in A$ and $bx \in A$. Therefore, $a \in [A : X]$ and $b \in [A : X]$. So, the subset $[A : X]$ is a strongly extensional consistent subset of S. (10) By proposition (8) of this theorem, we have $[A : x^2] \subseteq [A : x]$. On the other hand, let $a \in [A : x]$, i.e. let $ax \in A$. Then $(ax)^2 \in A$ because A is a semifilter of S, and therefore, we have $ax^2 \in A$ since A is a consistent subset of S. So, $a \in [A : x^2]$.□

In the next proposition we will give a very interesting property of the family $\{[A : x] : x \in A\}$ where A is a strongly extensional consistent filter of S. The proof of that theorem is technical.

Theorem 2.4 *Let A be a strongly extensional consistent subset of S. Then, A is a filter of S if and only if for every $x \in A$ we have $[A : x] = A$.*

Proof:

(1) Let A be a filter. Then

$a \in A \Longrightarrow ax \in A$ (because $x \in A$ and A is a filter in S)

$\Longleftrightarrow a \in [A : x]$.

(2) Let $[A : x] = A$ holds for every x in S. Let a and b be elements of S. We have

$$a \in A = [A : b] \wedge b \in A \Longrightarrow ab \in A.\Box$$

In the following theorem we analyze situation when the subset A is a primary subset of S.

Theorem 2.5 *Let A be a primary strongly extensional consistent subset of S.*

(1) If $x \in cr(A)$, then $[A : x] = A$.

(2) If $x \in A$, then $[A : x]$ is a primary subset of S and holds $cr([A : x]) = cr(A)$.

Proof: (1) Let $x \in cr(A)$ and let a be an arbitrary element of A. Then $ax \in A$, because A is a primary consistent subset of S. So, we have $a \in [A : x]$. Hence, out of fact $A \subseteq [A : x]$, we conclude $A = [A : x]$.

(2) At first, we conclude that $cr([A : x]) \subseteq cr(A)$. Let a be an arbitrary element of $cr(A) = cr(cr(A))$. Then, $(\exists n \in N)(a^n \in cr(A) \wedge x \in A)$. So, $(\exists n \in N)(xa^n \in A)$ and $(\exists n \in N)(a^n \in [A : x])$. Finally, we have $a \in cr([A : x])$. Therefore, we have $cr([A : x]) = cr(A)$. Further on , let a be an element of $cr(A) = cr([A : x])$ and b such that $bx \in A$. Then $abx \in A$ and, $ab \in [A : x]$. So, the quotient $[A : x]$ is a primary subset of S. □

In the following proposition we analyze set $[A : x]$ when strongly extensional consistent subset A is the union of filters under A:

Theorem 2.6 *Let $A = \cup F$ holds where the union is over all strongly extensional consistent filters under A. Then, $[A : x]$ is a potent semifilter of S .*

Proof: Let a be an arbitrary element of $[A : x]$, i.e. let $ax \in A = \cup F$. There exists a filter F under A such that $ax \in F$, i.e. $a \in [F : x] = F$ by Theorem 2. So, we have $a^n \in F = [F : x] \subseteq [\cup F : x]$ for every natural n. Therefore, the set $[A : x]$ is a potent semifilter of S. □

Corollary 2.7 *Let $A = \cup F$ holds where the union is over all strongly extensional consistent detachable filters under A. Then, $[\cup F : x] = \cup F$ holds .*

Proof:
$[\cup F : x] = F$. Indeed, let $a \in [\cup F : x]$. Then

$$\begin{aligned} a \in [\cup F_k : x] &\iff ax \in \cup F_k \\ &\iff (\exists F_k)(ax \in F_k) \\ &\iff (\exists F_k)(a \in [F_k : x]) \\ &\iff a \in \cup [F : x]. \end{aligned}$$

If $x \in F_k$, then $[F_k : x] = F_k$ (by Theorem 2.7). If $\neg(x \in F_k)$, then $[F_k : x] = \emptyset$ (by Theorem 2.5 (2)). Let $B = \{k : \neg(x \in F_k)\}$. Then we have

$$[\cup_{k\in A} F_k : x] = \cup_{k\in B} F_k. \square$$

If we try to make demands weak in the previous proposition, we have to give up from the demand $x \in \cup F$. Indeed, we have:

Theorem 2.8 *Let $A = \cup Q$ be a strongly extensional consistent subset of S which is union of primary strongly extensional consistent subsets of under A such that $\emptyset \neq [A : x] \neq A$. Then, there exists a filter F of S under A such that $x \bowtie F$.*

Proof: There exists an element y of A such that $y \bowtie [A : x]$. Hence, $y \in (A' : x)$, i.e. $x \in (A' : y) = [A : y]'$. Let $A = \cup Q$ holds where Q are primary strongly extensional consistent subsets of S under A. Then, there exists a primary subset Q of S such that $y \in Q$. Therefore, the subset $[Q : y]$ is a primary subset of S such that $cr([Q : x]) = cr(Q)$ by point (2) of Theorem 2.5. Since

$$cr(Q) = cr([Q : y]) \subseteq [Q : y] \subseteq [\cup Q : y] = [A : y],$$

we have $[A : y]' \subseteq cr(Q)'$. Therefore, $x \bowtie cr(Q)$. □

3 Constructions of maximal strongly extensional subsets

In the following several propositions (3.1- 3.7) we describe constructions (without the axiom of choose) the maximal strongly extensional consistent subset

$C(a)$ of S such that $a \bowtie C(a)$ (Theorem 3.5) and the maximal strongly extensional ideal $B(a)$ of S such that $a \bowtie B(a)$ (Theorem 3.7) for any a in S. These results themselves are interesting.

Lemma 3.1 *Let a and b be elements of S. Then the set $C_{(a)} = \{x \in S^1 : x \bowtie SaS\}$ is a consistent subset of S such that*:
(i) $a \bowtie C_{(a)}$;
(ii) $C_{(a)} \neq \emptyset \Longrightarrow 1 \in C_{(a)}$;
(iii) *Let a be an invertible element of S. Then* $C_{(a)} = \emptyset$;
(iv) $(\forall x, y \in S)(C_{(a)} \subseteq C_{(xay)})$;
(v) $C_{(a)} \cup C_{(b)} \subseteq C_{(ab)}$.

Proof:

$$\begin{aligned}(0)\ xy \in C_{(a)} &\Longleftrightarrow xy \bowtie SaS \\ &\Longrightarrow xy \bowtie SaSy \wedge xy \bowtie xSaS \\ &\Longrightarrow y \bowtie SaS \wedge x \bowtie SaS \\ &\Longleftrightarrow y \in C_{(a)} \wedge x \in C_{(a)}.\end{aligned}$$

(1) Let x be an arbitrary element of $C_{(a)}$. Then, $x \bowtie SaS$, and thus, $x \neq a$.

(2) Suppose that $C_{(a)} \neq \emptyset$. Then, there exists the element x of S such that $x \in C_{(a)}$. Thus, $x \cdot 1 \in C_{(a)}$ and we have $1 \in C_{(a)}$.

(3) Let a be an invertible element of S. Then, there exists the element b of S such that $ab = 1$. If $C_{(a)} \neq \emptyset$, then, by (2), $1 \in C_{(a)}$. Therefore, $a \in C_{(a)} \wedge b \in C_{(a)}$, what is impossible. So, $C_{(a)} = \emptyset$.

(4) Let x, y be arbitrary elements of S and let $u \bowtie SaS$. Then, $u \bowtie SxayS$. Therefore, $C_{(a)} \subseteq C_{(xay)}$.

(5) Out of (4), it immediately follows $C_{(a)} \subseteq C_{(ab)} \wedge C_{(b)} \subseteq C_{(ab)} \Longrightarrow C_{(a)} \cup C_{(b)} \subseteq C_{(ab)}$. □

Let a be an arbitrary element of a semigroup S with apartness. The consistent subset $C_{(a)}$ is called *principal consistent subset* of S generated by a. We introduce relation f, defined by $(a, b) \in f \Longleftrightarrow b \in C_{(a)}$. In the following theorem we will give some description of the relation f:

Lemma 3.2 *The relation f has the following properties*
(vi) *f is a consistent relation* ;
(vii) $(a, b) \in f \Longrightarrow (\forall x, y \in S)((xay, b) \in f)$;
(viii) $(a, b) \in f \Longrightarrow (\forall n \in N)((a^n, b) \in f)$;
(ix) $(\forall x, y \in S)((a, xby) \in f \Longrightarrow (a, b) \in f)$;
(x) $(\forall x, y \in S)\neg((a, xay) \in f)$.

Proof: (7) Let $(a, b) \in f$, i.e. let $b \in C_{(a)}$ and let x, y be arbitrary elements of S. Then $b \in C_{(xay)}$ by (iv). So, by definition of f, we have $(xay, b) \in f$.

(9) Let $(a, xby) \in f$ holds for some a, b, x, y in S. Then, $xby \in C_{(a)}$ and $b \in C_{(a)}$, because $C_{(a)}$ is a consistent subset of S, i.e. $(a, b) \in f$.

(10) Suppose that $(a, xay) \in f$ holds. Then, $(xay, xay) \in f$ by (vi), which is impossible. So, for every elements x and y $\neg((a, xay) \in f)$ holds. $\square$

We can construct the cotransitive relation $c(f) = \bigcap_{n \in N} {}^{n}f$ as cotransitive fulfillment of the relation f ([5],[6],[11]). As corollary of this theorem we have the following results:

Lemma 3.3 *The relation $c(f)$ satisfies the following properties*:
(xi) *$c(f)$ is a consistent relation on S* ;
(xii) *$c(f)$ is a cotransitive relation* ;
(xiii) $(\forall x, y \in S)((a, xay) \bowtie c(f))$;
(xiv) $(\forall n \in N)((a, a^n) \in c(f))$;
(xv) $(\forall x, y \in S)((a, b) \in c(f) \Longrightarrow (xay, b) \in c(f))$;
(xvi) $(\forall n \in N)((a, b) \in c(f) \Longrightarrow (a^n, b) \in c(f))$;
(xvii) $(\forall x, y \in S)((a, xby) \in c(f) \Longrightarrow (a, b) \in c(f))$.

Proof: (xi) and (xii) follows immediately from definition of $c(f)$.

(xiii) Let a, x, y be elements of a semigroup S and let (u, v) be an arbitrary element of $c(f)$. Then, $(u, a) \in c(f) \vee (a, xay) \in c(f) \vee (xay, v) \in c(f)$. Thus, $u \neq a \vee (a, xay) \in f \vee xay \neq v$. So, $(u, v) \neq (a, xay)$ because $\neg((a, xay) \in f)$ by (x).

(xiv) Follows from (xiii).

(xv) Let a, b, x, y be elements of S such that $(a, b) \in c(f)$. Then, $(a, xay) \in c(f) \vee (xay, b) \in c(f)$ and $(xay, b) \in c(f)$ by (xiii).

(xvi) Follows from (xv).

(xvii) $(a, xby) \in c(f) \Longrightarrow (a, b) \in c(f) \vee (b, xby) \in c(f)$
$\Longrightarrow (a, b) \in c(f)$ by (xiii). $\square$

Corollary 3.3.1 *The relation $c(f)$ is a positive potent-lower quasi-antiorder relation on S.*

Proof immediately follows from (xi) - (xiv) of the Lemma 3.3. $\square$

For an element a of a semigroup S and for $n \in N$ we introduce the following notations

$$A_n(a) = \{x \in S^1 : (a, x) \in {}^{n}f\},\ C(a) = \{x \in S^1 : (a, x) \in c(f)\}$$

$$B_n(a) = \{y \in S^1 : (y, a) \in {}^{n}f\},\ B(a) = \{y \in S^1 : (y, a) \in c(f)\}.$$

In the following two lemmas we will present some basic characteristics of these sets.

Lemma 3.4 *Let a and b be elements of a semigroup S. Then*:
(xvii) $A_1(a) = C_{(a)}$;
(xviii) $A_{n+1}(a) \subseteq A_n(a)$;
(xix) $A_{n+1}(a) = \{x \in S^1 : S^1 = A_n(a) \cup B_1(x)\}$;

(xx) $C(a) = \bigcap_{n\in N} A_n(a)$;
(xxi) $a \bowtie C(a)$;
(xxii) $C(a) \cup C(b) \subseteq C(ab)$;
(xxiii) *The set $C(a)$ is a strongly extensional consistent subset of S.*

Proof. (18)- (19) Let x be an arbitrary element of $A_{n+1}(a)$. Then, for every $t \in S$ we have $(a,t) \in {}^n f \vee (t,x) \in f$. So, $t \in A_n(a) \cup B_1(x)$, i.e. $S^1 = A_n(a) \cup B_1(x)$. As $(x,x) \bowtie f$, we have $(a,x) \in {}^n f$, i.e. $x \in A_n(a)$.

(21) If x is an element of $C(a)$, then $(a,x) \in c(f)$. Thus, $(a,x) \neq (a,a)$ because the relation $c(f)$ is cotransitive. Therefore, $x \neq a$.

(22) $x \in C(a) \vee x \in C(b) \Longleftrightarrow (a,x) \in c(f) \vee (b,x) \in c(f)$
$\Longrightarrow (a,ab) \in c(f) \vee (ab,x) \in c(f) \vee (b,ab) \in c(f) \vee (ab,x) \in c(f)$
$\Longrightarrow (ab,x) \in c(f)$ (by (xiii))
$\Longleftrightarrow x \in C(ab)$.

(23) Let x and y be arbitrary element of S such that $xy \in C(a)$, i.e. $(a,xy) \in c(f)$. Thus,out of $(a,xy) \in c(f)$ we have

$$((a,x) \in c(f) \vee (x,xy) \in c(f)) \wedge ((a,y) \in c(f) \vee (y,xy) \in c(f)).$$

Further on, it follows $x \in C(a) \wedge y \in C(a)$ (by (xiii)). Let $x \in A(a)$ and let y be an arbitrary element of S. Then, $(a,x) \in c(f)$. Thus, $(a,y) \in c(f) \vee (y,x) \in c(f)$. So, $y \in C(a) \vee y \neq x$. □

In the following theorem we give the proof (without the axiom of choose) that strongly extensional consistent subset $C(a)$ is the maximal for any a in S such that $a \bowtie C(a)$.

Theorem 3.5 *Let a be an element of a semigroup S. Then, the set $C(a)$ is the maximal strongly extensional consistent subset of S such that $a \bowtie C(a)$.*

Proof: Let T be a consistent strongly extensional subset of S such that $a \bowtie T$. Let t be an arbitrary element of T and let $u, v \in S$. Then, $t \neq uav \vee uav \in T$ holds. As $uav \in T$ is impossible because T is a consistent subset of S such that $a \in T$, then $A_1(a) = C(a) = \{x \in S^1 : x \bowtie SaS\} \supseteq T$. Let us assume that $A_n(a) \supseteq T$. Let t be an arbitrary element of T and let z be an arbitrary element of S. Then, $uzv \in T$ or $uzv \neq t$ for every u, v in S. Thus, $z \in T \subseteq A_n(a)$ or $uzv \neq t$ for every u, v in S. So, $z \in A_n(a) \vee t \in A_1(z)$. Therefore, $z \in A_n(a) \cup B_1(t)$. This means that $t \in A_{n+1}(a)$. Thus, by induction, we obtain that $A_n(a) \supseteq T$ for every $n \in N$, whence $C(a) \supseteq T$. Hence, $C(a)$ is the maximal strongly extensional consistent subset of S such that $a \bowtie C(a)$. □

Symmetrically, we have

Lemma 3.6 *Let a and b be elements of a semigroup S. Then*:
(a) $B(a) = \{y \in S^1 : b \in C_{(y)}\}$;
(b) $B_{n+1}(a) \subseteq B_n(a)$;

(c) $B_{n+1}(a) = \{y \in S^1 : S = B_n(a) \cup A_1(x)\}$;
(d) $B(a) = \bigcap_{n\in N} B_n(a)$;
(e) $a \bowtie B(a)$;
(f) $B(ab) \subseteq B(a) \cap B(b)$
(g) *The set* $B(a)$ *is a strongly extensional ideal of* S.

Proof.

(f) $x \in B(ab) \Longleftrightarrow ab \in A(x)$
$\Longrightarrow a \in A(x) \wedge b \in A(x)$
$\Longleftrightarrow x \in B(a) \wedge x \in B(b)$
$\Longleftrightarrow x \in B(a) \cap B(b)$.

(g) $x \in B(a) \vee y \in B(a) \Longleftrightarrow a \in A(x) \vee a \in A(y)$
$\Longleftrightarrow a \in A(x) \cup A(y)$
$\Longrightarrow a \in A(xy)$
$\Longleftrightarrow xy \in B(a)$.

Let x be an arbitrary element of S and let $y \in B(a)$. Then,
$(y, a) \in c(f) \Longrightarrow (y, x) \in c(f) \vee (x, a) \in c(f)$
$\Longrightarrow y \neq x \vee x \in B(a)$. □

Theorem 3.7 *Let* a *be an element of a semigroup* S. *Then, the set* $B(a)$ *is the maximal strongly extensional ideal of* S *such that* $a \bowtie B(a)$.

Proof: Let J be a strongly extensional ideal of S such that $a \bowtie J$. Then, $a \bowtie SJS$, i.e. $J \subseteq B_1(a)$. Let us assume that $J \subseteq B_n(a)$. Let z be an arbitrary element of S and let $y \in J \subseteq B_n(a)$. Then, $z \neq uyv$ or $z \in J$ for every u, v in S because J is a strongly extensional ideal of S. Thus, $z \in B_n(a) \vee z \in A_1(y)$, i.e. $y \in B_{n+1}(a)$. So, $J \subseteq B_{n+1}(a)$. By induction, we have $J \subseteq B(a)$. Therefore, the set $B(a)$ is the maximal strongly extensional ideal of S such that $a \bowtie B(a)$. □

Note that in the above theorem construction of the maximal strongly extensional ideal of S such that $a \bowtie B(a)$ is done without the axiom of choose.

The next proposition gives some interesting equalities about relation $c(f)$:

Theorem 3.8 *Let* S *is a semigroup with apartness. Then, the following conditions are equivalent*:
(1) $\forall(a, b \in S)((ab, a) \bowtie c(f) \vee (ab, b) \bowtie c(s))$;
(2) $(\forall a, b \in S)(A(ab) = A(a) \cup A(b))$;
(3) $(\forall a, b \in S)(B(ab) = B(a) \cap B(b))$;
(4) $A(a)$ *is a filter of* S *for every* a *in* S;
(5) $B(b)$ *is a completely prime ideal of* S *for every* b *in* S;
(6) $(\forall a, b \in S)((a, b) \bowtie c(f) \vee (b, a) \bowtie c(f))$.

Proof.

(6) $\Longrightarrow$ (1). This follows immediately.

(1) $\Longrightarrow$ (6). Let $a, b \in S$ and let (u, v) be an arbitrary element of $c(f)$. Then, $(u, a) \in c(f)$ or $(a, ab) \in c(f)$ or $(ab, b) \in c(f)$ or $(b, v) \in c(f)$. By (xi), (xiii) and by $(ab, b) \bowtie c(f)$, it follows that $(u, v) \neq (a, b)$. In a similar way we prove that by $(ab, a) \bowtie c(f)$, it follows that $(u, v) \neq (b, a)$.

(2) $\Longleftrightarrow$ (5). Let $xy \in B(b)$. Then, $b \in A(xy) = A(x) \cup A(y)$. Thus, $b \in A(x)$ or $b \in A(y)$. So, $x \in B(b)$ or $y \in B(b)$. If x be an arbitrary element of $A(ab)$, then $ab \in B(x)$ and $a \in B(x) \vee b \in B(x)$ because $B(x)$ is a completely prime ideal of S. Therefore, $x \in A(a) \cup A(b)$.

(3) $\Longleftrightarrow$ (4). Out of $(y \in B(a) \cap B(b) \Longleftrightarrow y \in B(a))$ and $(y \in B(b) \Longleftrightarrow a \in A(y))$ and $(b \in A(y) \Longleftrightarrow ab \in A(y) \Longleftrightarrow y \in B(ab))$, immediately follows the equivalence (3) $\Longleftrightarrow$ (4).

(4) $\Longrightarrow$ (1). Let (u, v) be an arbitrary element of $c(f)$ and let $a, b \in S$. Then, we have

$(u, v) \in c(f) \Longrightarrow ((u, ab) \in c(f) \vee (ab, a) \in c(f) \vee (a, v) \in c(f)) \wedge ((u, ab) \in c(f) \vee (ab, b) \in c(f) \vee (b, v) \in c(f))$

$\Longrightarrow (u \neq ab \vee a \in A(ab) \vee a \neq v) \wedge (u \neq ab \vee b \in A(ab) \vee b \neq v)$

$\Longrightarrow ((u, v) \neq (ab, a) \vee (u, v) \neq (ab, b)) \vee ab \in A(ab) \quad (A(ab)$ is a filter$)$

$\Longrightarrow (ab, a) \bowtie c(f) \vee (ab, b) \bowtie c(f)$.

(1) $\Longrightarrow$ (4). $x \in A(a) \wedge y \in A(a) \Longleftrightarrow (a, x) \in c(f) \wedge (a, y) \in c(f)$

$\Longrightarrow ((a, xy) \in c(f) \vee (xy, x) \in c(f)) \wedge ((a, xy) \in c(f) \vee (xy, y) \in c(f))$

$\Longrightarrow xy \in A(a)$.

(3) $\Longrightarrow$ (2). $x \in A(ab) \Longleftrightarrow (ab, x) \in c(f)$

$\Longrightarrow ((ab, a) \in c(f) \vee (a, x) \in c(f)) \wedge ((ab, b) \in c(f) \vee (b, x) \in c(f))$

$\Longrightarrow ab \in B(a) \cap B(b) = B(ab) \vee x \in A(a) \vee x \in A(b)$

$\Longrightarrow x \in A(a) \cup A(b)$.

(1) $\Longrightarrow$ (5). Suppose that $xy \in B(b)$. Thus, $(xy, b) \in c(f)$ implies $(xy, x) \in c(f) \vee (x, b) \in c(f)$ and $(xy, y) \in c(f) \vee (y, b) \in c(f)$. Therefore, by (1), we have $y \in B(b)$ or $x \in B(b)$. So, the ideal $B(b)$ is a completely prime ideal of S.

(3) $\Longrightarrow$ (6). Let (3) holds and let (u, v) be an arbitrary element of $c(f)$. Then, $(u, a) \in c(f) \vee (a, b) \in c(f) \vee (b, v) \in c(f)$ and $(u, b) \in c(f) \vee (b, a) \in c(f) \vee (a, v) \in c(f)$ for $a, b \in S$. If $(b, a) \in c(f) \wedge (a, b) \in c(f)$, i.e. if $b \in B(a)$ and $a \in B(b)$, then $ab \in B(a) \cap B(b) = B(ab)$ (because the sets $B(a)$ and $B(b)$ are ideals of S), what is impossible. So, $(u, v) \neq (b, a) \vee (u, v) \neq (a, b)$. □

Net us note that we do not know that the relation $c(f)$ is compatible with the semigroup operation on S, but we know that it is the maximal positive lower-potent quasi-antiorder relation on S. So, we do not know that the coequality $q = c(f) \cup (c(f))^{-1}$ is an anti-congruence on S.

4 Congruence and partial order generated by the quotient

In this part we shall analyze situation when $[A:x] \subseteq [A:y]$ for some elements x and y. At the end of this investigation we give a conclusion which follows from condition $[A:x] \subseteq [A:y]$

Theorem 4.1 *Let x and y be elements of semigroup S. If for every strongly extensional consistent subset A of S $[A:x] \subseteq [A:y]$ holds, then $C(x) \subseteq C(y)$.*

Proof. Let x and y be elements of S such that $[A:x] \subseteq [A:y]$. Since $C(y)$ is a strongly extensional consistent subset of S, by hypothesis, we have $[C(y):x] \subseteq [C(y):y]$. Since $y \bowtie C(y)$ must be $[C(y):y] = \emptyset$. Then, $[C(x):y] = \emptyset$ too, and $\neg(y \in C(x))$. Indeed, if $y \in C(x)$, then will be $1 \in [C(x):y] = \emptyset$, what it is impossible. Therefore, it must be $\neg(y \in C(x))$. Let u be an arbitrary element of $C(x)$. Since $C(x)$ is a strongly extensional subset of S, then $u \neq y \vee y \in C(x)$. So, $y \bowtie C(x)$ and $C(x)$ is a strongly extensional subset of S such that $y \bowtie C(x)$. Therefore, $C(x) \subseteq C(y)$ holds because $C(y)$ is the maximal strongly extensional subset of S such that $y \bowtie C(y)$. $\square$

Theorem 4.2 *Let A be a strongly extensional semifilter in S. Then, the relation e, defined by*

$$(x,y) \in e \Longleftrightarrow [A:x] = [A:y],$$

is a semillatice congruence on S.

Proof: Let A be a strongly extensional consistent subset of semigroup S. Then, the relation e on S, defined by $(x,y) \in e \Longleftrightarrow [A:x] = [A:y]$, has the following properties:

(1) The relation e is an equality relation on S.

(2) Let $(x,y) \in e$ and a be an arbitrary element of S. Then $[A:ax] = [A:ay]$. Indeed,

$$\begin{aligned} t \in [A:ax] &\Longleftrightarrow at \in [A:x] = [A:y] \\ &\Longleftrightarrow t \in [A:ay]. \end{aligned}$$

So, the relation e is a congruence on S.

(3) If A is a semifilter of S, then $[A:x] = [A:x^2]$, by (10) of Theorem 2.6. So, if A is a semifilter in S, then for all $x \in S$ we have $(x,x^2) \in e$. Further on, by commutatively of semigroup S, we have that e is a semilattice congruence on S. $\square$

Theorem 4.3 *Let A be a strongly extensional semifilter in S. Then, the relation α, defined by*

$$(x,y) \in \alpha \Longleftrightarrow (x,xy) \in e,$$

is a partial order on semigroup S compatible with the semigroup operation on S.

Proof: Let A be a semifilter of S and let we define relation α by the following equivalence $(x,y) \in \alpha \Longleftrightarrow (x,xy) \in e$, where $x,y \in S$. We conclude:
(i) $(x,x) \in \alpha$ because $(x,x^2) \in e$. So, α is a reflexive relation on S.

(ii) If $(x,y) \in \alpha$ and $(y,x) \in \alpha$, i.e. if $(x,xy) \in e$ and $(y,yx) \in e$, then $(x,y) \in e$. So, relation α is antisymmetric.

(iii) Let $(x,y) \in \alpha$ and $(y,z) \in \alpha$, i.e. let $(x,yx) \in e$ and $(y,yz) \in e$. Thus, $(xz,yxz) \in e$ and $(xy,xyz) \in e$, so, $(xz,xy) \in e$. Therefore, out of $(x,xy) \in e$ and $(xy,xz) \in e$ we conclude that $(x,xz) \in e$, i.e. the relation α is a transitive relation on S.

(iv) Let $(x,y) \in \alpha$, i.e. let $(x,xy) \in e$, i.e. let $[A:x] = [A:xy]$. Then, $[A:xz] = [A:xzyz]$. Indeed, at first, we have $[A:xzyz] \subseteq [A:xz]$ because the set A is a consistent subset of S. Secondly, let $t \in [A:xz]$, i.e. let $txz \in A$. Then, $(txz)(txz) \in A$ (A is a semifilter). Thus, $txztz \in [A:x] = [A:xy]$ and $txztzxy \in A$. Out of this we conclude that $txzyz \in A$ (because A is a consistent subset of S). So, $t \in [A:xzyz]$. Finally, we have that α is compatible with the semigroup operation. □

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